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Rapid Note

Invariable-profile wavefunctions and Brittingham's focus wave modes

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Abstract. A family of invariable-profile wavefunctions is constructed. The relations found describe both transient and steady-state waves. The Gauss and Bessel-Gauss focused waves of order m can be obtained from these steady-state waves *via* Bateman's transformation.

PACS. 03.40.Kf Waves and wave propagation: general mathematical aspects

In this paper we discuss interrelations between wavefunctions having invariable transverse profiles and Brittingham's focus wave modes of Gauss and Bessel-Gauss types [1,2]. We construct the explicit solutions of the initial-value problem to the inhomogeneous and homogeneous wave equations in cylindric coordinate system by using the specific transverse and angular distributions of the source or the boundary condition on the plane that starts at the fixed moment of time uniform motion along a straight line. As a result, we obtain the family of wavefunctions with invariable profiles, which describe both transient and steady-state wave processes. The latter enables us to get the focus weave modes of Gauss and Bessel-Gauss types of order m with the help of Bateman's transformation [3]. Here we apply the method elaborated by Hillion for construction of nondispersive solutions to the wave equations, in particular, the focus wave solution of the Gauss type of order zero [4,5].

We shall construct solutions of the inhomogeneous wave equation

$$\left(\nabla^2 - \frac{\partial^2}{\partial\tau^2}\right)\psi = \frac{4\pi}{c}j \tag{1}$$

where the source of the wave perturbation is

$$j = h(\tau)\delta(z-\beta\tau)f(\tau)\exp(im\varphi)R(\rho). \tag{2}$$

Here ρ, φ, z and $\tau = ct$ are the space and time variables, c is the wavefront velocity (that is, for electromagnetic waves the velocity of light), $v = \beta c$ is the velocity of the source plane, $\beta \in [0,1]$, $h(\tau)$ is the Heaviside function, $\delta(z-\beta\tau)$ is the Dirac function, $f(\tau)$ and $R(\rho)$ are continuous functions, and m is an integer.

The initial condition is

$$\psi \equiv 0 \qquad \tau < 0. \tag{3}$$

Representing the wavefunction ψ in the form

$$\psi = \exp(im\varphi)\psi_m(\rho, z, \tau) \tag{4}$$

we have from (1), (2), and (3)

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) - \frac{m^2}{\rho^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial\tau^2}\right]\psi_m = \frac{4\pi}{c}h(\tau)\delta(z-\beta\tau)f(\tau)R(\rho) \tag{5}$$

$$\psi_m \equiv 0 \quad \tau < 0.$$

(i) Let us now suppose that $R(\rho) = \rho^m$, $m \geq 0$. Then we get the function ψ_m in the form

$$\psi_m = \rho^m v(z,\tau) \tag{6}$$

where $v(z,\tau)$ is the solution of the initial-value problem to 1D wave equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial\tau^2}\right)v = \frac{4\pi}{c}h(\tau)\delta(z-\beta\tau)f(\tau),$$
$$v \equiv 0 \quad \tau < 0. \tag{7}$$

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Then we have in the space-time domain $z - \beta\tau < 0$

$$v = -\frac{2\pi}{c\sqrt{1-\beta^2}} \int_0^{\tau_\beta + z_\beta} d\tau' f\left(\frac{\tau'}{\sqrt{1-\beta^2}}\right) \quad (8)$$

where $\tau_\beta = \frac{\tau - \beta z}{\sqrt{1-\beta^2}}$, $z_\beta = \frac{z-\beta\tau}{\sqrt{1-\beta^2}}$, and find the function ψ by using the relations (4) and (6). For finite ρ one can get finite solutions of the homogeneous wave equation for $\beta \in [0, 1]$ which satisfies the same conditions on the plane $z - \beta\tau = 0$ by differentiating the found wavefunctions with respect to the variable z. Summing up the results we can write wavefunctions with the transverse profiles ρ^m as

$$\psi = \psi_{0m} \exp(im\varphi)\rho^m v(\tau + z), \qquad z - \beta\tau < 0 \quad (9)$$

where v is arbitrary function of the variable $\tau + z$ and ψ_{0m} is a constant. It should be noticed that the transient wave process does not accompany formation of the above waves.

(ii) We obtain the other type of wavefunctions with invariable profiles when the transverse distribution of the source (2) (or the boundary condition) is described by the Bessel function of the first kind, $R(\rho) = J_m(a\rho)$, a is constant. Then we find a solution of the problem (1), (2), and (3) in the form

$$\psi = \exp(im\varphi)J_m(a\rho)u(z,\tau) \quad (10)$$

where

$$u = -\frac{2\pi}{c\sqrt{1-\beta^2}} \int_0^{\tau_\beta + z_\beta} d\tau' f\left(\frac{\tau'}{\sqrt{1-\beta^2}}\right) \times J_0\left(a\sqrt{(\tau_\beta - \tau')^2 - z_\beta^2}\right), \qquad \tau_\beta + z_\beta > 0 \quad (11)$$

is a solution of the initial-value problem of 1D telegraph equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \tau^2} - a^2\right) u = \frac{4\pi}{c} h(\tau)\delta(z - \beta\tau) f(\tau), \qquad u \equiv 0 \quad \tau < 0. \quad (12)$$

In the case of time dependence of the source $f(\tau) = \exp(ik\tau)$, k is constant, we represent expression (11) in terms of Lommel's functions of two variables $U_n(w,\mu)$ or $V_n(w,\mu)$ (see [6] for details). Let us suppose, for example, that $k > a > 0$. Then one can write from (11) for $w_\pm/\mu < 1$

$$u = -\frac{2\pi}{c\sqrt{1-\beta^2}\sqrt{k_\beta^2 - a^2}} \times \left[U_1(w_+,\mu) - U_1(w_-,\mu) + i\left(U_2(w_+,\mu) - U_2(w_-,\mu)\right)\right] \quad (13)$$

where $k_\beta = k/\sqrt{1-\beta^2}$, $\mu = a\sqrt{\tau_\beta^2 - z_\beta^2}$, $w_\pm = \left(k_\beta \pm \sqrt{k_\beta^2 - a^2}\right)(\tau_\beta + z_\beta)$, while for $w_+/\mu > 1$ and $w_-/\mu < 1$

$$\begin{aligned} u = &-\frac{2\pi}{c\sqrt{1-\beta^2}\sqrt{k_\beta^2-a^2}} \\ &\times \left[V_1(w_+,\mu) + iV_0(w_+,\mu) - U_1(w_-,\mu) - iU_2(w_-,\mu)\right] \\ &+ \frac{2\pi}{c\sqrt{1-\beta^2}\sqrt{k_\beta^2-a^2}} \exp\left[\frac{i}{2}\left(\left(w_+ + \frac{\mu^2}{w_+}\right) + \pi\right)\right] \\ = &\, u^t + u^s. \end{aligned} \quad (14)$$

The second term u^s, describing the steady-state solution, is formed by means of terms $U_n(w_+, z)$, $n = 1, 2$ only. We can find finite solution of the homogeneous equation by differentiating the expression (11) with respect to the variable z [7]. Summing up the results, we obtain the second type of the steady-state wavefunctions with invariable profiles

$$\psi = \psi_{0m} \exp(im\varphi)\, J_m(a\rho) \exp\left[\frac{i}{2}\left(\left(k_\beta + \sqrt{k_\beta^2 - a^2}\right)(\tau_\beta + z_\beta) + \left(k_\beta - \sqrt{k_\beta^2 - a^2}\right)(\tau_\beta - z_\beta)\right)\right] \quad (15)$$

which exist in the domain

$$\tau_\beta \frac{\left[a^2 - \left(k_\beta + \sqrt{k_\beta^2 - a^2}\right)^2\right]}{\left[a^2 + \left(k_\beta + \sqrt{k_\beta^2 - a^2}\right)^2\right]} < z_\beta < 0. \quad (16)$$

When the source (the boundary) plane travels with the velocity of light, we obtain from (15) and (16)

$$\psi(\beta = 1) = \psi_{0m} \exp(im\varphi) J_m(a\rho) \times \exp\left[\frac{i}{2}\left(k(\tau + z) + \frac{a^2}{k}(\tau - z)\right)\right] \quad (17)$$

$$\tau\left(a^2 - k^2\right)/\left(a^2 + k^2\right) < z < \tau.$$

One can verify by substitution that wavefunctions (15) and (17) with the complex parameters a and k satisfy the homogeneous wave equation. Formation of the u^s-type solution with the complex k and imaginary a is investigated in details in [6].

We obtain wave modes akin to the Gauss and Bessel-Gauss types from the wavefunctions having invariable profiles by means of Bateman's transform [3]

$$\psi(\rho_0, \varphi, z_0, \tau_0) \to \tilde{\psi} = \frac{1}{z_0 - \tau_0}\psi\left(\frac{\rho_0}{z_0 - \tau_0}, \varphi, \frac{r_0^2 - \tau_0^2 - 1}{2(z_0 - \tau_0)}, \frac{r_0^2 - \tau_0^2 + 1}{2(z_0 - \tau_0)}\right) \quad (18)$$

where $\rho_0 = \rho\lambda$, $z_0 = z\lambda$, and $\tau_0 = \tau\lambda$ are the dimensionless variables, $\lambda > 0$ is constant, and $r_0^2 = \rho_0^2 + z_0^2$.

We get the explicit expression for nondispersive wavefunctions of order m from (9) in the form

$$\tilde{\psi}_m = \psi_{0m} \exp(im\varphi) \frac{\rho^m}{\lambda(z-\tau)^{m+1}} v(z+\tau+\frac{\rho^2}{z-\tau}), \quad z-\tau \neq 0. \quad (19)$$

Hence making v the exponential function one can obtain the focus wave modes of the Gauss type. Also, wavefunction (9) has the invariable profile if v is an arbitrary function of the variable $\tau - z$. In this case one finds the nondispersive wavefunctions applying transform (18) in which z is replaced by $-z$.

Note that one can get solution (19) of the homogeneous wave equation by separating variables φ and $\rho/(\tau - z)$ and bearing in mind that the function $\frac{1}{z-\tau} v(z+\tau+\frac{\rho^2}{z-\tau})$ is an axisymmetric solution of the wave equation.

The explicit relations for the wavefunctions which are akin to the Bessel-Gauss focus wave modes of order m may be obtained from (15) or (17). Using transform (18) we get from expression (17)

$$\tilde{\psi}_m = \psi_{0m} \exp(im\varphi) \frac{1}{\lambda(z-\tau)} J_m \left(\frac{a\rho}{\lambda(z-\tau)} \right) \times \exp\left[\frac{i}{2} k \left(z+\tau+\frac{\rho^2+(a/k\lambda)^2}{z-\tau} \right) \right] \quad z-\tau \neq 0. \quad (20)$$

Replacing z, τ, k, and a^2/k by z_β, τ_β, $k_\beta + \sqrt{k_\beta^2 - a^2}$, and $k_\beta - \sqrt{k_\beta^2 - a^2}$ we have wavefunction constructed from (15) by the same method.

Supposing that $\psi_{0m} \sim 1/a^m$ and taking the limit $|a| \to 0$ we obtain wavefunction (19) where $v = \exp\left[ik \left(z+\tau+\frac{\rho^2}{z-\tau} \right) \right]$.

In the general case of the orthogonal cylindric coordinates x_1, x_2, z, one can separate the transverse variables x_1, x_2 representing the solution of the wave equation in the form $\psi = X(x_1, x_2) Z(z, \tau)$ where the function $Z(z, \tau)$ is the solution of the telegraph equation of the type (11), (14). Then using Bateman's transform it is possible, in principle, to obtain the wavefunction $\tilde{\psi}$ that contains the factor $\frac{1}{z-\tau} \exp\left[\alpha_1 \left(z+\tau-\frac{\rho^2+\alpha_2^2}{z-\tau} \right) \right]$ (here constants $\alpha_{1,2}$ should be in agreement with the transverse distribution of the source or the boundary conditions), which is characteristic for the focused waves. However, this possibility requires an individual investigation.

Note that obtained solutions of the scalar wave equation can be applied to the description of the electromagnetic waves. It is clear that the Cartesian components of the electric field strength and the magnetic induction vectors satisfy the wave equation. In some cases electromagnetic field vectors are expressed in terms of scalar functions with the help of the electric and magnetic one-component Hertz vectors, which in cylindric coordinates have the form $\mathit{\Pi} = \mathbf{e}_z \mathit{\Pi}$ and $\mathit{\Pi}^* = \mathbf{e}_z \mathit{\Pi}^*$ where functions $\mathit{\Pi}$ and $\mathit{\Pi}^*$ are solutions of the scalar wave equation [8].

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